out by the ancient philosophers but they were the source of much of the speculation in mediaeval times, when mathematical and philosophical thought were closely allied. One writer, John Campanus of Novava, thought that the princiciple of the golden section descended from the gods. Keplertcompared it to a precious stone, and called it *proportio divina*, but not *proportio* or *sectio aurea*. The latter name has originated since his time.

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

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(Continued from the July-August Number.)

The central-polar equation of the Cassinian Oval may be written

$$r^4-(2c^2\cos 2\theta)r^2=m^4-c^4 \ldots . (1).$$

$$\therefore \cos 2\theta=\frac{r^4-(m^4-c^4)}{2c^2r^2}, \text{ and } \sin 2\theta=\sqrt{\left(\frac{4c^4r^4-[r^4-(m^4-c^4)]}{4c^4r^4}\right)}.$$

$$\therefore \mathsf{P}=8m^2\int_b^a \frac{r^2dr}{\sqrt{\{4c^4r^4-[r^4-(m^4-c^4)]^2\}}}$$

$$=8m^2\int_b^a \frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}} \ldots . (2).$$

Reducing (2) under the supposition that

$$r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi,$$

$$\mathsf{P}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{r}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}} \ldots . (j),$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2(1-2\sin^2\phi)]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2\cos 2\phi]^{\frac{1}{4}}} \ldots . (3).$$

Let $2\phi=\psi$, and make $2m^2c^2/(m^4+c^4)=C$; then, after obvious transformations, (3) gives

$$\mathsf{P}=\frac{2m^2}{[m^4+c^4]^{\frac{1}{4}}}\int_0^{\pi}\frac{d\psi}{[1+C\cos\psi]^{\frac{1}{2}}}\dots(4).$$

After expanding (4) into a series of not less than two dozen terms, and observing that the negative terms of the series will *vanish* on taking the integral limits, we obtain a series expressing the perimeter of the Cassinian Oval. Since $m^2=5$ and $c^2=4$; that is, since the semi-axes of the Cassinian Oval in consideration are 3 and 1 linear units, we have $C=\frac{4}{4}\frac{0}{1}$. After a rather laborious calculation, we find $\mathsf{P}=14.9831+$ linear units. On page 223 of the July-August MONTHLY, $C=\frac{8}{8}\frac{0}{1}$; and four terms of that resulting series give a perimeter ($\mathsf{P}=12.7329+$ linear units) too small by $2\frac{1}{4}$ linear units. Since the *moduli* of these functions are almost unity, the resulting series will not converge rapidly; and with this same trouble, it must be remembered, *M.* Legendre also had to contend. Possibly some of the talented readers of the MONTHLY will succeed in expanding (j), or (4), into a *rapidly-converging* series.

After performing certain rather elaborate transformations of *premises approximative in derivation*, we deduce the following two remarkable formulae for the perimeter of the Cassinian Oval:

II. Transforming the Cartesian equation of the Cassinian Oval by the formulae, $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$r^2=\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta\dots(1).$$

$$\therefore\ rdr=\frac{-c^2[\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta]\sin 2\theta d\theta}{\surd(m^4-c^4\sin^2 2\theta)}\dots(\alpha),$$

$$\text{and}\left(\frac{dr}{d\theta}\right)^2=\frac{c^4[\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta]\sin^2 2\theta}{m^4-c^4\sin^2 2\theta}\dots(\beta).$$

$$\therefore\ \mathsf{P}=4m^2\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta}{m^4-c^4\sin^2 2\theta}\right)}d\theta\dots(2).$$

Put $(c^2/m^2)^2=C^2$; then (2) can easily be transformed into

$$\mathsf{P}=4m\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\surd(1-C^2\sin^2 2\theta)+C\surd(1-\sin^2 2\theta)}{1-C^2\sin^2 2\theta}\right)}d\theta$$

$$=4m\surd(1+C)\int_0^{\frac{1}{2}\pi}\sqrt{\left[1+\left(\frac{2(1+C^2)-(2+C)}{2}\right)\sin^2 2\theta\right]}d\theta$$

$$=4m\sqrt{\left(\frac{(1+C)[2(1+C^2)-C]}{2}\right)}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1-\left(1-\frac{2}{2(1+C^2)-C}\right)\sin^2\phi\right]}d\phi,$$

$$=\tfrac{3}{2}\pi m\sqrt{\left(\frac{(1+C)[2(1+C^2)-C]}{2}\right)}\left[1-\Sigma\left(\frac{1.3.5\dots(2n-1)}{2.4.6\dots 2n}\right)^2\frac{M^{2n}}{2n-1}\right];$$

of which elliptic function M is the *modulus*, and n may have all consecutive integral values from *unity* to *infinity*. When $m^2=5$ and $c^2=4$, $\mathsf{P}=14.9652$.

III. From the Cartesian equation of the Cassinian Oval, we deduce

$$y^2=\surd(m^4+4c^2x^2)-(c^2+x^2)\ldots.(3),$$

an equation which gives all the *real* points of the Oval in consideration.

$$\therefore\left(\frac{dy}{dx}\right)^2=\frac{[4c^4-4c^2\surd(m^4+4c^2x^2)+m^4+4c^2x^2]x^2}{(m^4+4c^2x^2)[\surd(m^4+4c^2x^2)-(c^2+x^2)]}\ldots.(\gamma).$$

Representing the semi-axis major of the Cassinian Oval by $a,=\surd(m^2+c^2)$, we have

$$\mathsf{P}=4\left(\frac{2c^2-m^2}{m^2\surd(m^2-c^2)}\right)\int_0^a\sqrt{\left(\frac{m^4(m^2-c^2)}{(2c^2-m^2)^2}+x^2\right)}dx$$

$$=2\left[\left(\frac{\surd[m^4(m^2-c^2)+(m^2+c^2)(2c^2-m^2)^2]}{m^2}\right)\sqrt{\left(\frac{m^2+c^2}{m^2-c^2}\right)}\right.$$

$$+\frac{m^2\surd(m^2-c^2)}{2c^2-m^2}$$

$$\left.\log\left(\frac{(2c^2-m^2)\surd(m^2+c^2)+\surd[m^4(m^2-c^2)+(m^2+c^2)(2c^2-m^2)^2]}{m^2\surd(m^2-c^2)}\right)\right],$$

$$=\tfrac{16}{9}\left[\frac{3\surd(106)}{5}+\tfrac{5}{3}\log\left(\frac{9+\surd(106)}{5}\right)\right]=14.9833,$$

when $m^2=5$ and $c^2=4$.

[To be continued.]

POSTULATE II. OF EUCLID'S ELEMENTS.

By Professor JOHN N. LYLE, Ph. D., Westminister College, Fulton, Missouri.

"Let it be granted that a terminated straight line may be produced to any length in a straight line."

Euclid lays down the statement just quoted as his second postulate regulative of geometrical constructions. Wherever in unbounded space any point may be located to which a straight line has been extended, Euclid assumes that the straight line may be lengthened out beyond that point.

Riemann assumes that every straight line is finite in length, and if extended will ultimately return to the starting point.

If a straight line that is produced from a given point eventually returns to the same point, Euclid's postulate 2 is false.

On the other hand, if the second postulate of Euclid is true, the Rie-